Mapping class groups
Problem sheet 1

Lent 2021

1. Give an example of a surface S of finite type and a self-diffeomorphism ϕ of S which is homotopic to id_S but not isotopic to id_S.

2. Let D^2 be the closed unit disc, and let $D^2_* = D^2 \smallsetminus \{0\}$. Prove that every self-homeomorphism of D^2_* extends to a self-homeomorphism of D^2 that fixes 0.

3. Let $A \in SL_2(\mathbb{R})$ be a non-identity matrix, and let ϕ_A be the corresponding element of $PSL_2(\mathbb{R}) \equiv \mathrm{Isom}^+(\mathbb{H}^2)$. Prove that:

   (i) if $|\mathrm{tr}\, A| > 2$ then ϕ_A is hyperbolic;

   (ii) if $|\mathrm{tr}\, A| = 2$ then ϕ_A is parabolic;

   (iii) if $|\mathrm{tr}\, A| < 2$ then ϕ_A is elliptic.

   Here, as usual, $\mathrm{tr}\, A$ denotes the *trace* of A.

4. (a) Geodesic lines in $\mathbb{H}^2$ that do not meet in $\overline{\mathbb{H}}^2$ are called *ultraparallel.* Prove that two ultraparallel geodesic lines γ_1, γ_2 are joined by a unique common perpendicular.

   (b) Prove that the endpoints of this perpendicular are the unique points that achieve the minimal distance between γ_1 and γ_2.

   (c) Let ϕ be a hyperbolic isometry of $\mathbb{H}^2$, with translation length τ. Prove that if x is not on the axis of $\mathbb{H}^2$ then $d(x, \phi(x)) > \tau$.

5. Recall that $S_{0,n,0}$ is the sphere with n punctures, and $S_{0,0,b}$ is the sphere with b boundary components.

   (a) Sketch the construction of a hyperbolic structure $S_{0,n,0}$ for suitable n.

   (b) Sketch the construction of a hyperbolic structure on $S_{0,0,b}$ for suitable b.

6. Let S be a hyperbolic surface, and let α be a closed curve on S which is homotopic into a puncture. A *horocycle* is a circle in the hyperbolic plane (in either the upper half-plane or disc model) which meets the boundary in exactly one point. Prove that, after a homotopy, α has a lift $\tilde{\alpha}$ which is a horocycle.

7. Let S be a hyperbolic surface and α a closed curve that is not homotopic to a point. Prove that the centraliser $C_{\pi_1 S}(\alpha)$ is cyclic and that the centre $Z(\pi_1 S)$ is trivial.

8. Let α be a closed curve on the 2-torus T^2. Prove that α is homotopic to a simple closed curve if and only if α represents a primitive element of $\pi_1 T^2$.

9. Recall that the fundamental group of the 2-torus T^2 is isomorphic to $\mathbb{Z}^2$. Suppose the simple closed curve α corresponds to (a,b) and β corresponds to (c,d). Prove that $i(\alpha,\beta) = |ad - bc|$.

10. Prove the Euclidean case of the bigon criterion. That is, let S be a surface of finite type with $\chi(S) = 0$ and let α, β be transverse, essential, simple closed curves on S. Prove that if α and β are not in minimal position then they form a bigon.

11. Prove that every surface S has a collection of essential closed curves and proper arcs that satisfy the hypotheses of the Alexander method: that is, there are no bigons, no annuli and no triangles.

12. (a) Exhibit a homotopy equivalence between the 3-punctured sphere $S_{0,3,0}$ and the punctured torus $S_{1,1,0}$.

    (b) Show that there are self-homotopy-equivalences of $S_{0,3}$ that are not homotopic to homeomorphisms.